Math 6082 2025-02-07

§ 3.4 Ergodicity.

Def. A measure preserving system (MPS) $(X, \mathcal{B}, \mu, T)$ is called *ergodic* if

$$B \in \mathcal{B} \text{ with } T^{-1}B = B \implies \mu(B) = 0 \text{ or } 1.$$

We simply say T is ergodic if $(X, \mathcal{B}, \mu, T)$ is ergodic.

In another word, an ergodic MPS has no non-trivial invariant subset: any invariant subset equals either X or ϕ a.e. (subject to a set of zero measure).

For $A, B \subset X$, write $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Thm 1 The following are equivalent:

(i) T is ergodic.

(ii) The only member $B \in \mathcal{B}$ with $\mu(T^{-1}B \triangle B) = 0$ are those with $\mu(B) = 0$ or $\mu(B) = 1$.

(iii) For any $A \in \mathcal{B}$ with $\mu(A) > 0$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} T^{-n}A\right) = 1.$$

(iv) For any $A, B \in \mathcal{B}$ with $\mu(A) > 0$, $\mu(B) > 0$, there exists $n \in \mathbb{N}$ such that

$$\mu(T^{-n}A \cap B) > 0.$$

Pf. (i) $\Rightarrow$ (ii).

Let $B \in \mathcal{B}$ with $\mu(B \triangle T^{-1}B) = 0$.

We will construct $B_\infty \in \mathcal{B}$ with $B_\infty = T^{-1}B_\infty$ such that

$$\mu(B \triangle B_\infty) = 0.$$

Notice that for any $n \geq 1$,

$$\begin{aligned} T^{-n}B \triangle B &\subset \bigcup_{i=0}^{n-1} \left(T^{-i-1}B \triangle T^{-i}B\right) \\ &= \bigcup_{i=0}^{n-1} T^{-i}(T^{-1}B \triangle B). \end{aligned}$$

By the invariance of μ,

$$\mu(T^{-n}B \triangle B) = 0.$$

Let

$$B_\infty = \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} T^{-i}B$$

Note that $T^{-1}B_\infty = \bigcap_{i=1}^{\infty} \bigcup_{n=i+1}^{\infty} T^{-i}B = B_\infty$.

Observe that

$$m\left(B \triangle \left(\bigcup_{n=i}^{\infty} T^{-i}B\right)\right) \leq \sum_{n=i}^{\infty} m(B \triangle T^{-i}B)$$

$$= 0$$

Hence letting $i \to \infty$,

$$m(B \triangle B_\infty) = 0.$$

(ii) $\Rightarrow$ (iii): Let $A \in \beta$ with $\mu(A) > 0$.

Set $C = \bigcup_{i=1}^{\infty} T^{-i}(A)$.

Then $T^{-1}C = \bigcup_{i=2}^{\infty} T^{-i}(A) \subset C$.

However, since $\mu(T^{-1}C) = \mu(C)$, it follows that

$$\mu(T^{-1}C \triangle C) = \mu(C \setminus T^{-1}C) = 0.$$

By (ii), $\mu(C)=0$ or $\mu(C)=1$.

But $\mu(C) \geq \mu(T^{-1}A) = \mu(A) > 0$, so $\mu(C)=1$,

i.e. $\mu\left(\bigcup_{i=1}^{\infty} T^{-i}A\right) = 1$.

(iii) $\Rightarrow$ (iv):

Let $A, B \in \mathcal{B}$ with $\mu(A), \mu(B) > 0$.

Then $\mu\left(\bigcup_{n=1}^{\infty} T^{-n}A\right) = 1$.

Hence $\mu\left(\left(\bigcup_{n=1}^{\infty} T^{-n}A\right) \cap B\right) = \mu(B) > 0$.

i.e. $\mu\left(\bigcup_{n=1}^{\infty} (T^{-n}A \cap B)\right) > 0$.

Hence $\exists\ n \geq 1$ s.t. $\mu(T^{-n}A \cap B) > 0$.

(iv) $\Rightarrow$ (i). Suppose on the contrary T is not ergodic.

Then $\exists$ B with $T^{-1}B = B$, but $0 < \mu(B) < 1$

Let $A = X \setminus B$.

Then $T^{-n}A = A$ and $\mu(A) > 0$.

Hence $T^{-n}A \cap B = A \cap B = \phi$.

Thus $\mu(T^{-n}A \cap B) = 0$ for all $n \geq 1$, leading to a contradiction. $\boxed{\ }$

Thm 2. Let $(X, \mathcal{B}, \mu, T)$ be a MPS. Then the following are equivalent:

(i) T is ergodic.

(ii) when f is measurable, $f(TX) = f(x)$ for all $x \in X$, then f is constant a.e.

(iii) when f is measurable, $f(Tx) = f(x)$ a.e, then f is constant a.e.

(iv) when $f \in L^2(\mu)$, $f(Tx) = f(x)$ for all $x \in X$, then f is constant a.e.

(v) $f \in L^2(\mu)$, $f(TX) = f(x)$ a.e $\Rightarrow$ f is constant a.e.

Pf. For brevity we only prove (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let $B \in \mathcal{B}$ with $T^{-1}B = B$.

Set $f = \chi_B$. By (ii), f is constant a.e.

$\Rightarrow \mu(B) = 0$ or 1.

(i) $\Rightarrow$ (ii). Let f be measurable with $f \circ T = f$.

For any $n \geq 1$, $j \in \mathbb{Z}$, s.t.

$$A_{n,j} = \left\{ x : \frac{j}{n} \leq f(x) < \frac{j+1}{n} \right\}.$$

Then $T^{-1} A_{n,j} = A_{n,j}$ for all n, j.

Hence $\mu(A_{n,j}) = 0$ or 1.

Since $\{A_{n,j}\}_{j \in \mathbb{Z}}$ is a partition of X,

$\exists$ $j_n \in \mathbb{Z}$ such that $\mu(A_{n,j_n}) = 1$.

Now take

$$B = \bigcap_{n=1}^{\infty} A_{n,j_n}.$$

Then $\mu(B) = 1$. But for all $x \in B$, $f(x) = \lim_{n\to\infty} \frac{j_n}{n}$.

$\blacksquare$

Example 3. (Rotation on the circle).

Let $X = \mathbb{R}/\mathbb{Z}$, $\alpha \in (0,1)$ and

$$Tx = x + \alpha \pmod 1.$$

Let μ be the Haar measure on $\mathbb{R}/\mathbb{Z}$.

Then T is ergodic $\Leftrightarrow$ α is irrational.

Pf. First assume $\alpha = \frac{p}{q} \in \mathbb{Q}$.

Define $f(x) = qx \pmod 1$.

Then $f(Tx) = f(x)$ for all x.

But f is not const a.e.

Hence T is not ergodic.

Next assume α is irrational.

Let $f \in L^2(\mu)$ with $f \circ T = f$.

Let $f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$ be the Fourier series of f.

$$f(Tx) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x} \cdot e^{2\pi i n \alpha}$$

Hence $a_n = a_n e^{2\pi i n \alpha}$ for all $n \in \mathbb{Z}$.

It implies $a_n = 0$ for all $n \neq 0$.

Hence f is const. a.e.

Example 4. (Doubling map on the circle)

$X = \mathbb{R}/\mathbb{Z}$, $Tx = 2x \pmod 1$, μ — Haar measure.

Then T is ergodic.

. Similarly, one use Fourier series.

Indeed, let $f \in L^2(\mu)$ with $f \circ T = f$.

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

$$f(Tx) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i 2n x}$$

Hence $\begin{cases} a_{2n} = a_n \\ a_{2n+1} = 0 \end{cases}$.

which implies that $a_n = 0$ if $n \neq 0$,

and thus, f is const a.e.

Example 5. (Shift map).

Let $\Sigma^{\mathbb{N}} = \prod_{n=1}^{\infty} \{1, \cdots, m\}$ be one-sided shift.

Let $(p_1, \cdots, p_k)$ be a prob. vector with $p_i > 0$.

Define μ on Σ by

$$\mu([x_1 \cdots x_n]) = p_{x_1} \cdots p_{x_n}.$$

Then m is σ-invariant. Moreover, m is ergodic.